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A System of Partial Differential Equations Arising in Electrophotography

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We consider the process which takes place in the photocopy machine as soon as the document to be photocopied is exposed to light. This process is modeled by a coupled system of one elliptic equation for the voltage V and one parabolic equation for the charge density ρ with small diffusion ε. The leading coefficient in the equation for V has some discontinuities with jump whose size is a nonlinear functional (with memory) of ρ and V. We prove that there exists a unique global solution and then study the vector field ∇V and the charge density ρ as $\varepsilon \to 0$.
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1. THE MODEL

Electrophotography produces images using light and electric charges. The process is much faster than traditional photography (which uses light and chemical processes). Photocopy machines work on the principle of electrophotography. After you insert a document to be copied, you push the button for "print." The pre-charged film on the photoconductor drum is then immediately exposed to flash, and a photodischarge process takes place, resulting in an electrical image of the document on the photoconductor drum. It is precisely this process that we wish to study in this paper; for more details on electrophotography we refer the reader to the books [5, 10–12].

The part of the photoconductor drum with which we are concerned consists of two layers (which for simplicity we take to be horizontal slabs). The bottom one $\{0 < y < h_2\}$ is called the generating layer, and the top one $\{h_2 < y < h_1\}$ is called the transport layer (see [7, Chap. 17]). The exposure is coming from above the transport layer.

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Photons from the exposed light travel through the transport layer and penetrate the generating layer, where they create electrons and holes. The electrons are trapped at the bottom $\{y=0\}$ of the generating layer which had been originally positively charged. On the other hand the holes move in the upward direction into and through the transport layer, and end up by discharging their positive charge at the top $\{y=h_1\}$. It is this discharge that makes up the electrical image of the document.

Let

$$D_2=\{(x,y); 0<x<a, 0<y<h_2\} \qquad \text{(generating layer)},$$
$$D_1=\{(x,y); 0<x<a, h_2<y<h_1\} \qquad \text{(transport layer)},$$
$$D_0=\{(x,y); 0<x<a, h_1<y<\infty\} \qquad \text{(air)}$$

and set

$$\kappa=\begin{cases}\kappa_2 & \text{in } D_2\\ \kappa_1 & \text{in } D_1\\ 1 & \text{in } D_0,\end{cases}$$

where κ_i is the electric conductivity in the medium D_i ($\kappa_0=1$). We introduce

ρ = density of holes,

V = electric field potential (= voltage),

$\mathbf{E}$ = electric field ($\mathbf{E}=-\nabla V$).

Then ρ, V satisfy the following system of differential equations

$$\frac{\partial\rho}{\partial t}-\varepsilon\Delta\rho=\mu\nabla\cdot(\rho\nabla V) \qquad \text{in int } \overline{D_2\cup D_1} \qquad (\varepsilon>0), \tag{1.1}$$

$$\nabla\cdot(\kappa\nabla V)=-\frac{4\pi}{\varepsilon_0}\rho \qquad \text{in } D_2\cup D_1\cup D_0 \tag{1.2}$$

with jump relations

$$[\kappa\partial_y V]_{y=h_2-0}^{y=h_2+0}=0, \tag{1.3}$$

$$[\kappa\partial_y V]_{y=h_1-0}^{y=h_1+0}=-\frac{4\pi}{\varepsilon_0}\sigma, \tag{1.4}$$

and

$$\frac{\partial\sigma}{\partial t}=[\mu\rho\partial_y V+\varepsilon\rho_y]_{y=h_1-0}; \tag{1.5}$$

the diffusion coefficient ε is positive but very small, μ (the mobility of holes) and ε_0 (the permittivity of the vacuum) are positive constants, and σ is surface charge density on $\{y=h_1\}$. Equation (1.1) is the law of conservation of charge $\partial\rho/\partial t+\operatorname{div} J=0$ where $J=-\mu\rho\nabla V-\varepsilon\nabla\rho$ is the current density, and (1.5) is the relation $\partial\sigma/\partial t=J_2$ (= the y-component of J). We impose the boundary conditions

$$\rho_y(x, 0, t)=\rho_y(x, h_1, t)=0 \qquad \text{(holes do not leak out of } \{0\leqslant y\leqslant h_1\}), \tag{1.6}$$

$$V=0 \qquad \text{for} \quad y=0, \tag{1.7}$$

$$|\nabla V|\to 0 \qquad \text{if} \quad y\to\infty, \tag{1.8}$$

and periodic boundary conditions at $x=0$, $x=a$:

$$V(0, y, t)=V(a, y, t), \qquad V_x(0, y, t)=V_x(a, y, t), \tag{1.9}$$

$$\rho(0, y, t)=\rho(a, y, t), \qquad \rho_x(0, y, t)=\rho_x(a, y, t). \tag{1.10}$$

Finally, we prescribe initial conditions

$$\sigma(x, 0)=\sigma_0(x), \qquad \sigma_0(0)=\sigma_0(a) \tag{1.11}$$

and

$$\rho(x, y, 0)=f(x, y), \qquad f(0, y)=f(a, y). \tag{1.12}$$

It is assumed that

$$\sigma_0<0, \qquad f\geqslant 0; \tag{1.13}$$

typically

$$f(x, y)\simeq\begin{cases} c & \text{if} \quad a_1<x<a_2, 0<y<b_1 \\ 0 & \text{otherwise,}\end{cases}$$

where $0<a_1<a_2<a$, $0<b_1<h_2$, and c is positive and of order of magnitude 1.

The system (1.1)–(1.12) is a coupled system of elliptic and nonlinear parabolic equations; if $\varepsilon=0$ the parabolic equation becomes hyperbolic. The system is similar to the space charge problem studied in [4]. The condition (1.4) (which accounts for surface charge accumulation on $\{y=h_1\}$) makes the present problem more complicated than the one studied in [4]. We also note that the geometry in [4] is entirely different from the geometry in the present problem, and that the system in [4] is time-independent. These differences affect significantly the type of physical properties that one wishes to establish.

In Sections 1–7 we prove existence and uniqueness of a solution ρ_ε, V_ε of (1.1)–(1.12). The rest of the paper is concerned with physical properties of the solution. In Section 8 we prove that if $|\sigma_0|$ is large then for some small time T the vector field ∇V_ε is such that, uniformly in ε, $V_{\varepsilon,y}$ is negative and large and it controls $|V_{\varepsilon,x}|$ (except near $y=h_1$). This means that the positive charge should move upward towards $y=h_1$ where it will be deposited. Indeed, this conclusion (in a somewhat weak form) is established in Section 9; more precisely, we prove that if we have "good enough" initial field (i.e., if f is as in (1.14)) and if the initial surface charge is strong enough then we get a "good" crisp image.

2. Reformulation of the Problem

For simplicity we take

$$\mu=1, \qquad \frac{4\pi}{\varepsilon_0}=1. \tag{2.1}$$

Set

$$\begin{gathered}\Omega_\lambda=\{-\infty<x<\infty, 0<y<\lambda\}, \qquad \lambda>0,\\ Q_2=\Omega_{h_2}, \qquad Q_1=\Omega_{h_1}\backslash\overline{\Omega_{h_2}}, \qquad Q_0=\Omega_\infty\backslash\overline{\Omega_{h_1}}.\end{gathered} \tag{2.2}$$

Instead of studying (1.1)–(1.12) in $\{0<x<a\}$ with periodic boundary conditions at $x=0$, $x=a$, it will be more convenient to study the system for all $x\in\mathbb{R}^1$ and (at the very end) impose the condition of periodicity in x. Thus we consider

$$\rho_t=\varepsilon\Delta\rho+\nabla V\cdot\nabla\rho-\frac{\rho^2}{\kappa} \qquad \text{in } \Omega_{h_1}\times(0,\infty), \tag{2.3}$$

$$\begin{aligned}\nabla\cdot(\kappa\nabla V)=-\rho \qquad &\text{in } (Q_2\cup Q_1\cup Q_0)\times(0,\infty),\\ &\text{where } \rho\equiv 0 \text{ in } Q_0\times(0,\infty),\end{aligned} \tag{2.4}$$

$$[\kappa\partial_y V]_{y=h_2-0}^{y=h_2+0}=0, \tag{2.5}$$

$$[\kappa\partial_y V]_{y=h_1-0}^{y=h_1+0}=-\sigma_0(x)+\int_0^t \rho(x,h_1)\,V_y(x,h_1-0,s)\,ds, \tag{2.6}$$

$$\rho_y=0 \qquad \text{if } y=0 \text{ or } y=h_1, \tag{2.7}$$

$$V(x,0,t)=0, \tag{2.8}$$

$$|\nabla V|\to 0 \qquad \text{if } y\to\infty, \tag{2.9}$$

$$\rho(x,y,t) \text{ and } V(x,y,t) \text{ are periodic in } x \text{ of period } a. \tag{2.10}$$

Finally we prescribe initial conditions

$$\rho(x, y, 0) = f(x, y), \qquad (x, y) \in Q_1 \cup Q_2 \tag{2.11}$$

and assume that

$$\begin{aligned} &f \text{ is in } C^{2+\alpha} \qquad \text{for some } 0<\alpha<1, \\ &f \geqslant 0 \end{aligned} \tag{2.12}$$

$$\sigma_0 \text{ is in } C^{2+\alpha}, \qquad \sigma_0(x) \leqslant -N < 0 \tag{2.13}$$

$$f(x, y), \sigma_0(x) \text{ are periodic in } x \text{ of period } a. \tag{2.14}$$

In Section 7 we prove that the system (2.3)–(2.14) has a unique global classical solution. Various auxiliary results are proved in Sections 3–6.

3. A Diffraction Problem

In this section we consider an auxiliary diffraction problem

$$\nabla(\kappa \nabla u) = -\rho \qquad \text{in } Q_2 \cup Q_1 \cup Q_0, \tag{3.1}$$

$$[\kappa u_y]_{y=h_1-0}^{y=h_1+0} = g(x), \tag{3.2}$$

$$[\kappa u_y]_{y=h_2-0}^{y=h_2+0} = 0, \tag{3.3}$$

$$u(x, 0) = 0, \tag{3.4}$$

where

$$\begin{aligned} &\rho \in C^\alpha(\mathbb{R}_2^+), \qquad g \in C^{1+\alpha}(\mathbb{R}^1) \qquad (\mathbb{R}_+^2 \equiv \mathbb{R}^1 \times [0, \infty)) \\ &\rho(x, y) = 0 \qquad \text{if } y \geqslant N \text{ (for some } N > h_1), \end{aligned} \tag{3.5}$$

and

$$|\rho|_{L^\infty(R_+^2)} < \infty, \qquad |g|_{L^\infty(\mathbb{R}^1)} < \infty. \tag{3.6}$$

Lemma 3.1. *If u is a bounded solution of (3.1)–(3.4) then*

$$|u|_{L^\infty(\mathbb{R}_+^2)} < C(|\rho|_{L^\infty(\mathbb{R}_+^2)} + |g|_{L^\infty(\mathbb{R}^1)}), \tag{3.7}$$

where C is a constant depending only on N, h_2, h_1, and κ_1, κ_2.

Proof. Let

$$g_i(x, y) = \begin{cases} N - h_i, & y > N \quad (N > h_1) \\ y - h_i, & h_i < y \leqslant N \\ C_1(y - h_i), & y \leqslant h_i \end{cases}$$

and

$$g_N(x, y) = \begin{cases} \frac{1}{2}|\rho|_{L^\infty} N^2, & y \geqslant N \\ (Ny - \frac{1}{2}y^2)\,|\rho|_{L^\infty}, & y < N. \end{cases}$$

Let $X_0 = (0, -1)$ and, for any $R > N+2$, set

$$v_R(x, y) = A\,\frac{\log|X - X_0|}{\log R},$$

where $X = (x, y)$. The constants C_1, C_2, and A are positive and will be determined later on. We shall compare u with the function

$$w(x, y) = (|g|_{L^\infty} + |\rho|_{L^\infty})\left(\sum_{i=1}^{2} g_i(x, y)\right) + g_N(x, y) + v_R(x, y) + C_3$$

in $\Omega_R = \{X = (x, y);\ |X - X_0| < R\} \cap \mathbb{R}^2_+$ where C_3 is a suitable positive constant.

Notice that

$$\Delta w = -|\rho|_{L^\infty} \leqslant \Delta u \qquad \text{in } \Omega_R \cap \{y \neq N, y \neq h_1, y \neq h_2\}, \tag{3.8}$$

and w is continuous in $\overline{\Omega_R}$. We examine the discontinuities of w_y on $\{y = N\}$, $\{y = h_1\}$, and $\{y = h_2\}$.

On $\{y = N\}$,

$$[w_y] = w_y(x, N+0) - w_y(x, N-0) = -2(|g|_{L^\infty} + |\rho|_{L^\infty}) < 0. \tag{3.9}$$

On $\{y = h_1\}$,

$$\begin{aligned}[\kappa w_y] &= w_y(x, h_1 + 0) - \kappa_1 w_y(x, h_1 - 0) \\ &= [(1-\kappa_1) + (1 - \kappa_1 C_1)](|g|_{L^\infty} + |\rho|_{L^\infty}) + (1-\kappa_1)(N - h_1)\,|\rho|_{L^\infty} \\ &\quad + (1-\kappa_1)\frac{A}{\log R}\,\frac{h_1+1}{x^2 + (h_1+1)^2} \\ &\leqslant [-\kappa_1 C_1 + |2-\kappa_1| + |1-\kappa_1|\,(N-h_1)](|g|_{L^\infty} + |\rho|_{L^\infty}) \\ &\quad + |1-\kappa_1|\,\frac{A}{\log R}.\end{aligned}$$

Hence, for any $A > 0$, if we choose R such that

$$\frac{A}{\log R} \leqslant (|g|_{L^\infty} + |\rho|_{L^\infty}) \tag{3.10}$$

and if C_1 is defined by

$$\kappa_1 C_1 = |2-\kappa_1| + |1-\kappa_1| \, (N-h_1+1)+2 \tag{3.11}$$

then

$$[\kappa w_y]_{h_1-0}^{h_1+0} < -|g|_{L^\infty}. \tag{3.12}$$

Similarly, on $y = h_2$,

$$\begin{aligned}
[\kappa w_y] &= \kappa_1 w_y(x, h_2+0) - \kappa_2 w_y(x, h_2-0) \\
&= [(\kappa_1 - \kappa_2 C_2) + (\kappa_1-\kappa_2)\, C_1](|g|_{L^\infty} + |\rho|_{L^\infty}) \\
&\quad + (\kappa_1-\kappa_2)(N-h_2)\,|\rho|_{L^\infty} + (\kappa_1-\kappa_2)\frac{A}{\log R}\frac{h_2+1}{x^2+(h_2+1)^2} \\
&\leqslant [-\kappa_2 C_2 + \kappa_1 + (C_1+N-h_2+1)\,|\kappa_1-\kappa_2|](|g|_{L^\infty}+|\rho|_{L^\infty})
\end{aligned}$$

by (3.10). For a suitable choice of C_2, depending only on C_1, κ_i, h_2, we then get

$$[\kappa w_y]_{h_2-0}^{h_2+0} < 0. \tag{3.13}$$

Comparing (3.9), (3.12), (3.13) with (3.2), (3.3) we conclude that

$$[\kappa(w-u)_y]_{y=\lambda-0}^{y=\lambda+0} < 0 \qquad \text{for} \quad \lambda = N, h_1, h_2. \tag{3.14}$$

We next evaluate w on $\partial\Omega_R$. On $y=0$,

$$w(x,0) = (-C_1 h_1 - C_2 h_2)(|g|_{L^\infty} + |\rho|_{L^\infty}) + v_R(x,0) + C_3 \geqslant 0$$

if

$$C_3 = (C_1 h_1 + C_2 h_2)(|g|_{L^\infty} + |\rho|_{L^\infty}).$$

On the other hand, if $|X - X_0| = R$,

$$w(x,y) > A - C(|\rho|_{L^\infty} + |g|_{L^\infty}) > |u|_{L^\infty},$$

provided

$$A = |u|_{L^\infty(\mathbb{R}_2^+)} + C(|\rho|_{L^\infty} + |g|_{L^\infty}).$$

With this choice of A we then have that $w-u \geqslant 0$ on $\partial\Omega_R$. In view of (3.8), $w-u$ is superharmonic in Ω_R, off the lines $y=N$, $y=h_1$, and $y=h_2$. We claim that

$$w \geqslant u \qquad \text{in} \quad \Omega_R.$$

Indeed, otherwise $w-u$ must take negative minimum at either $y=N$, $y=h_1$, or $y=h_2$. Suppose for definiteness that it takes it at (x_0, h_1). Then

$$\partial_y(w-u)(x_0, h_1+0) \geqslant 0,$$
$$\partial_y(w-u)(x_0, h_1-0) \leqslant 0$$

and we conclude that

$$[\kappa \partial_y(w-u)]_{h_1-0}^{h_1+0} \geqslant 0,$$

a contradiction to (3.14).

We have thus proved that $u \leqslant w$ in Ω_R. If we apply this result to $-u$, we obtain

$$|u(x, y)| \leqslant w(x, y) \qquad \text{for any} \quad (x, y) \in \Omega_R.$$

Fixing (x, y) and letting $R \to \infty$ we get

$$|u(x, y)| \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty}),$$

and the lemma follows.

COROLLARY 3.2. *There exists at most one bounded solution of* (3.1)–(3.4).

Indeed, if u_1, u_2 are two bounded solutions the $v \equiv u_1 - u_2$ is a bounded solution with $g=0$, $\rho=0$. By (3.7), $v \equiv 0$.

COROLLARY 3.3. *If ρ and g are periodic in x of period a, then so is the bounded solution of* (3.1)–(3.4).

Indeed, since $u(x, y)$ and $u(x+a, y)$ are both bounded solutions of (3.1)–(3.4), Corollary 3.2 implies that $u(x+a, y) = u(x, y)$.

4. GRADIENT ESTIMATES FOR THE DIFFRACTION PROBLEM

In this section we estimate the gradient ∇u of the solution of (3.1)–(3.4) in terms of $|\rho|_{L^\infty}$, $|g|_{L^\infty}$. To this end we introduce the auxiliary problem

$$\Delta u = -\rho \qquad \text{in} \quad \mathbb{R}_+^2 \setminus \{y=\lambda\}, \tag{4.1}$$

$$\beta_1 u_y(x, \lambda+0) - \beta_2 u_y(x, \lambda-0) = g(x), \qquad \beta_1 \neq \beta_2, \tag{4.2}$$

$$u(x, 0) = 0, \tag{4.3}$$

where ρ, g satisfy (3.5), (3.6), and λ, β_1, β_2 are positive constants.

Introduce the Green function for the Laplacian in the half-plane

$$G(x, y; \xi, \eta)=\frac{1}{4\pi}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2} \qquad (y>0, \eta>0).$$

LEMMA 4.1. *For any $f \in C^\alpha(\mathbb{R}^1)$ there exists a unique $\tilde{f} \in C^\alpha(\mathbb{R}^1)$ such that the function*

$$v(x, y)=\int_{-\infty}^{\infty} G(x, y; \xi, \lambda)\, \tilde{f}(\xi)\, d\xi \tag{4.4}$$

has the properties:

(i) $v(x, 0)=0$,

(ii) $\Delta v=0$ *in* $\mathbb{R}^2_+\setminus\{y=\lambda\}$,

(iii) $\beta_1 v_y(x, \lambda+0)-\beta_2 v_y(x, \lambda-0)=f(x)$;

moreover,

$$|\tilde{f}|_{L^\infty(\mathbb{R}_1)} \leqslant \frac{2}{\beta_1+\beta_2}\frac{|f|_{L^\infty(\mathbb{R}^1)}}{1-|\mu|}, \tag{4.5}$$

where $\mu=(\beta_1-\beta_2)/(\beta_1+\beta_2)$.

Proof. For any $\tilde{f} \in C^\alpha(\mathbb{R}^2_+)$, the function v defined by (4.4) satisfies (i), (ii). Further, if $y \neq \lambda$,

$$\begin{aligned} v_y(x, y)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(y-\lambda)\,\tilde{f}(\xi)}{(x-\xi)^2+(y-\lambda)^2}\,d\xi-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(y+\lambda)\,\tilde{f}(\xi)}{(x-\xi)^2+(y+\lambda)^2}\,d\xi \\ &\equiv J_1+J_2. \end{aligned}$$

J_1 can be written in the form

$$\begin{aligned} J_1&=\frac{\tilde{f}(x)}{2\pi}\int_{-\infty}^{\infty}\frac{(y-\lambda)\,d\xi}{(x-\xi)^2+(y-\lambda)^2}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(y-\lambda)(\tilde{f}(\xi)-\tilde{f}(x))}{(x-\xi)^2+(y-\lambda)^2}\,d\xi \\ &\equiv J_{11}+J_{12} \end{aligned}$$

and

$$\begin{aligned} J_{11}&=\frac{\tilde{f}(x)}{2\pi}\arctan\frac{\xi-x}{y-\lambda}\bigg|_{-\infty}^{\infty} \\ &=\begin{cases} -\frac{1}{2}\tilde{f}(x) & \text{if } y<\lambda \\ \frac{1}{2}\tilde{f}(x) & \text{if } y>\lambda, \end{cases} \\ |J_{12}|&\leqslant\frac{1}{2\pi}|\tilde{f}|_{C^\alpha}\int_{-\infty}^{\infty}\frac{|y-\lambda|\,|\xi-x|^\alpha}{(x-\xi)^2+(y-\lambda)^2}\,d\xi \\ &\leqslant C\,|\tilde{f}|_{C^\alpha}\,|y-\lambda|^\alpha\to 0 \qquad \text{if } y\to\lambda. \end{aligned}$$

It follows that

$$\beta_1 v_y(x, \lambda+0) - \beta_2 v_y(x, \lambda-0)$$
$$= \frac{\beta_1+\beta_2}{2} \bar{f}(x) - \frac{\beta_1-\beta_2}{\pi} \int_{-\infty}^{\infty} \frac{\lambda \bar{f}(\xi)}{(x-\xi)^2+4\lambda^2} \, d\xi.$$

Hence (iii) means that $\bar{f}$ is a solution of the integral equation

$$\bar{f}(x) = \mu \int_{-\infty}^{\infty} K(x, \xi) \, \bar{f}(\xi) \, d\xi + \frac{2}{\beta_1+\beta_2} f(x), \tag{4.6}$$

where

$$K(x, \xi) = \frac{1}{\pi} \frac{2\lambda}{(x-\xi)^2+4\lambda^2}.$$

Clearly

$$K(x, \xi) > 0, \qquad \int_{-\infty}^{\infty} K(x, \xi) \, d\xi = 1.$$

In deriving (4.6) we have assumed that $\bar{f} \in C^\alpha$. We shall now prove that (4.6) has a unique solution in

$$C_b = \{k; k \text{ bounded and continuous in } \mathbb{R}^1\};$$

since the integral on the right-hand side of (4.6) is in $C^\alpha(\mathbb{R}^1)$ if $\bar{f} \in L^\infty(\mathbb{R}^1)$, it then follows that the solution $\bar{f}$ of (4.6) is in $C^\alpha(\mathbb{R}^1)$, and, with the exception of (4.5), all the assertions of the lemma follow.

In order to solve (4.6) in C_b, let

$$C_{b,M} = \{k \in C_b, |k|_{L^\infty} \leqslant M\},$$

where

$$M \geqslant \frac{2}{\beta_1+\beta_2} \frac{|f|_{L^\infty}}{1-|\mu|}. \tag{4.7}$$

For any $k \in C_{b,M}$ define the mapping

$$Tk(x) = \mu \int_{-\infty}^{\infty} K(x, \xi) \, k(\xi) \, d\xi + \frac{2}{\beta_1+\beta_2} f(x).$$

Then

$$|Tk|_{L^\infty} \leqslant |\mu| \, |k|_{L^\infty} \int_{-\infty}^{\infty} K(x, \xi) \, d\xi + \frac{2}{\beta_1+\beta_2} |f|_{L^\infty}$$
$$\leqslant |\mu| \, M + (1-|\mu|) \, M = M,$$

i.e., T maps $C_{b,M}$ into itself. It is also clear that

$$|Tk_1 - Tk_2|_{L^\infty} \leqslant |\mu|\ |k_1 - k_2|_{L^\infty},$$

i.e., T is a contraction. It follows that T has a unique fixed point in $C_{b,M}$ for any M large; consequently there is a unique solution $\tilde{f}$ of (4.6) in C_b, and

$$|\tilde{f}|_{L^\infty} \leqslant M.$$

Since we can choose the equality sign in (4.7), the assertion (4.5) follows.

LEMMA 4.2. *There exists a unique bounded solution of* (4.1)–(4.3) *and, for y near λ,*

$$|u_y(x, y)| \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty}), \tag{4.8}$$

$$|u_x(x, y)| \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty})(1 + |\log|y-\lambda|\,|). \tag{4.9}$$

Proof. The function

$$\begin{aligned} u(x, y) = &\int_0^\infty \int_{-\infty}^\infty G(x, y; \xi, \eta)\, \rho(\xi, \eta)\, d\xi\, d\eta \\ &+ \int_{-\infty}^\infty G(x, y; \xi, \lambda)\, \tilde{f}(\xi)\, d\xi \equiv u_1 + u_2, \end{aligned} \tag{4.10}$$

where $\tilde{f}$ is given by Lemma 4.1 with

$$f(x) = g(x) - (\beta_1 - \beta_2) \frac{\partial u_1}{\partial y}(x, \lambda), \tag{4.11}$$

is a solution of (4.1)–(4.3). One can easily check that u is bounded and, in fact,

$$|u|_{L^\infty} \leqslant C(|\rho|_{L^\infty} + |\tilde{f}|_{L^\infty}).$$

By Corollary 3.2, u is the unique bounded solution of (4.1)–(4.3). It remains to establish the estimates (4.8), (4.9).

For y near λ, $y \neq \lambda$,

$$\begin{aligned} u_y = &\frac{1}{2\pi} \int_0^N \int_{-\infty}^\infty \frac{2\eta[y^2 - \eta^2 - (x-\xi)^2]\, \rho(\xi, \eta)}{[(x-\xi)^2 + (y-\eta)^2][(x-\xi)^2 + (y+\eta)^2]}\, d\xi\, d\eta \\ &+ \frac{1}{2\pi} \int_{-\infty}^\infty \left[\frac{y-\lambda}{(x-\xi)^2 + (y-\lambda)^2} - \frac{y+\lambda}{(x-\xi)^2 + (y+\lambda)^2}\right] \tilde{f}(\xi)\, d\xi \\ = &\,\partial_y u_1 + \partial_y u_2. \end{aligned} \tag{4.12}$$

Note that for y near λ,

$$(x-\xi)^2+(y+\eta)^2 \geqslant c(1+(x-\xi)^2), \qquad c>0,$$
$$|y^2-\eta^2-(x-\xi)^2| \leqslant C[|y-\eta|+(x-\xi)^2].$$

To estimate $\partial_y u_1$ we write

$$\begin{aligned}|\partial_y u_1| \leqslant |\rho|_{L^\infty} C \bigg\{ &\int_{|X-Y|<1} \frac{dY}{|X-Y|} \\ &+\int_{\substack{|X-Y|>1\\ |y-\eta|>1/2}} \frac{[|y-\eta|+(x-\xi)^2]\, dY}{[(x-\xi)^2+(y-\eta)^2][1+(x-\xi)^2]} \\ &+\int_{\substack{|x-\xi|>1/2\\ |y-\eta|<1/2}} \frac{dY}{1+(x-\xi)^2}\bigg\},\end{aligned}$$

where $X=(x, y)$, $Y=(\xi, \eta)$. Each of the last three integrals is bounded by a constant independent of (x, y). Hence

$$|\partial_y u_1|_{L^\infty} \leqslant C\, |\rho|_{L^\infty}. \tag{4.13}$$

Next, from the form of $\partial_y u_2$ in (4.12),

$$|\partial_y u_2|_{L^\infty} \leqslant C\, |\tilde{f}|_{L^\infty} \leqslant C_1\, |f|_{L^\infty} \qquad \text{(by (4.5))},$$

where f is given by (4.11). Recalling (4.13) we obtain the bound

$$|\partial_y u_2|_{L^\infty} \leqslant C(|\rho|_{L^\infty}+|g|_{L^\infty})$$

and therefore, together with (4.13), (4.12), the assertion (4.8) follows.

To prove (4.9) we compute, analogously to (4.12),

$$\begin{aligned}u_x &= \frac{1}{2\pi} \int_0^N \int_{-\infty}^{\infty} \frac{4(x-\xi)\, y\eta\rho(\xi, \eta)\, d\xi\, d\eta}{[(x-\xi)^2+[y-\eta)^2][(x-\xi)^2+(y+\eta)^2]} \\ &\quad +\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4(x-\xi)\, y\lambda\tilde{f}(\xi)\, d\xi}{[(x-\xi)^2+(y-\lambda)^2][(x-\xi)^2+(y+\lambda)^2]} \\ &= \partial_x u_1+\partial_x u_2.\end{aligned} \tag{4.14}$$

We split the integral of $\partial_x u_1$ into $|X-Y|<1$ and $|X-Y|>1$; the first integrand is bounded by

$$C\, |\rho|_{L^\infty} \cdot \frac{1}{|X-Y|}$$

and the second integrand is bounded by

$$C\,|\rho|_{L^\infty}\cdot\frac{1}{1+(x-\xi)^2}.$$

It follows that

$$|\partial_x u_1|\leqslant C\,|\rho|_{L^\infty}. \tag{4.15}$$

On the other hand

$$\begin{aligned}|\partial_x u_2| &\leqslant C\,|\tilde{f}|_{L^\infty}\left\{\int_{|x-\xi|>1}\frac{|x-\xi|\;y\lambda}{|x-\xi|^4}\,d\xi+\int_{|x-\xi|<1}\frac{|x-\xi|\;y\lambda}{(x-\xi)^2+(y-\lambda)^2}\,d\xi\right\}\\ &\leqslant C\,|\tilde{f}|_{L^\infty}\left\{1+\int_0^1\frac{s\,ds}{s^2+(y-\lambda)^2}\right\}\\ &\leqslant C\,|\tilde{f}|_{L^\infty}\,\{1+|\log|y-\lambda|\,|\}.\end{aligned}$$

Using this and (4.15) in (4.14), and recalling that

$$|\tilde{f}|_{L^\infty}\leqslant C(|g|_{L^\infty}+|\partial_y u_1(\cdot,\lambda)|_{L_\infty}),$$

the assertion (4.9) follows.

We now return to the diffraction problem (3.1)–(3.4) and establish gradient estimates, assuming that u is a bounded solution.

LEMMA 4.3. *If $y\geqslant 2N$ then the bounded solution u of* (3.1)–(3.4) *satisfies:*

$$|\nabla u(x,y)|\leqslant\frac{C}{y}(|\rho|_{L^\infty}+|g|_{L^\infty}). \tag{4.16}$$

Proof. Set $f(x)=u(x,N)$, $\bar{u}(x,y)=u(x,y+N)$. Then

$$\Delta\bar{u}=0 \quad \text{in} \quad \mathbb{R}^1\times(0,\infty),$$
$$\bar{u}(x,0)=f(x)$$

and, by Lemma 3.1,

$$|f|_{L^\infty}\leqslant C(|\rho|_{L^\infty}+|g|_{L^\infty}). \tag{4.17}$$

From the representation formula

$$\bar{u}(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-\xi)^2+y^2}f(\xi)\,d\xi$$

we obtain

$$\partial_y \bar{u}(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\frac{1}{(x-\xi)^2 + y^2} - \frac{2y^2}{[(x-\xi)^2 + y^2]^2} \right] f(\xi)\, d\xi$$

so that

$$\begin{aligned} |\partial_y \bar{u}(x, y)| &\leqslant C\, |f|_{L^\infty} \int_{-\infty}^{\infty} \frac{y^2}{((x-\xi)^2 + y^2)^2}\, d\xi \\ &\leqslant C\, |f|_{L^\infty} \int_{-\infty}^{\infty} \frac{d\xi}{(x-\xi)^2 + y^2} \leqslant \frac{C}{y} (|\rho|_{L^\infty} + |g|_{L^\infty}), \end{aligned}$$

where (4.17) was used.

Similarly

$$\begin{aligned} \left| \partial_x \bar{u}(x, y) \right| &= \frac{1}{\pi} \left| \int_{-\infty}^{\infty} \frac{2(x-\xi)\, y}{((x-\xi)^2 + y^2)^2} f(\xi)\, d\xi \right| \\ &\leqslant C\, |f|_{L^\infty} \int_{-\infty}^{\infty} \frac{d\xi}{(x-\xi)^2 + y^2} \leqslant \frac{C}{y} (|\rho|_{L^\infty} + |g|_{L^\infty}), \end{aligned}$$

and (4.16) follows.

LEMMA 4.4. *For y near h_1, the bounded solution of* (3.1)–(3.4) *satisfies:*

$$|u_y(x, y)| \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty}), \tag{4.18}$$

$$|u_x(x, y)| \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty})[1 + |\log |y - h_1| \,|]. \tag{4.19}$$

Proof. Let $\eta > 0$ be such that

$$\eta \leqslant \tfrac{1}{4} \min\{h_2, h_1 - h_2, N - h_1\}$$

and let $\chi \in C^\infty(\mathrm{R}^1)$ be such that $\chi(x) = 1$ if $|x| \leqslant \eta$, $\chi(x) = 0$ if $|x| \geqslant 2\eta$. Let

$$v(x, y) = u(x, y)\, \tilde{\chi}(y), \qquad \tilde{\chi}(y) = \chi(y - h_1).$$

Then

$$\Delta v = -\frac{\rho}{\kappa} \tilde{\chi} + u_y \tilde{\chi}' + u \tilde{\chi}'' \equiv \tilde{\rho} \qquad \text{in} \quad \mathbb{R}^2_+ \setminus \{y = h_1\}$$

and

$$\begin{aligned} [\kappa v]_{h_1 - 0}^{h_1 + 0} &= g, \\ v(x, 0) &= 0. \end{aligned}$$

Applying Lemma 4.2 with $\beta_1 = 1$, $\beta_2 = \kappa_1$, $\lambda = h_1$, we obtain

$$|v_y(x, y)| \leqslant C(|\tilde{\rho}|_{L^\infty} + |g|_{L^\infty}),$$
$$|v_x(x, y)| \leqslant C(|\tilde{\rho}|_{L^\infty} + |g|_{L^\infty})(1 + |\log |y - h_1| |)$$

for y near h_1. Since by Lemma 3.1 and the Schauder elliptic estimates,

$$|u_y \tilde{\chi}'| \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty}),$$

we clearly have

$$|\tilde{\rho}|_{L^\infty} \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty})$$

and the assertions (4.18), (4.19) follow for y near h_1.

LEMMA 4.5. *For y near h_2 the bounded solution of* (3.1)–(3.4) *satisfies:*

$$|\nabla u(x, y)| \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty}). \tag{4.20}$$

Proof. The proof of (4.18) extends also to y near h_2. Thus it remains to prove that

$$|u_x(x, y)| \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty}) \tag{4.21}$$

for y near h_2. Set $v(x, y) = u(x, y)\,\hat{\chi}(y)$ where $\hat{\chi}(y) = \chi(y - h_2)$, χ as in the proof of Lemma 4.4. Then

$$\Delta v = \hat{\rho}, \qquad |\hat{\rho}|_{L^\infty} \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty}).$$

The functions

$$u_1(x, y) = v(x, y + h_2), \qquad u_2(x, y) = v(x, -y + h_2)$$

are defined for $0 < y < \eta$ and satisfy:

$$u_1(x, 0) = u_2(x, 0),$$
$$\kappa_1 \partial_y u_1(x, 0) + \kappa_2 \partial_y u_2(x, 0) = 0.$$

By L^p-elliptic estimates [1, Th. 10.2] we deduce that

$$|u_i|_{W^{2,p}} \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty}) \qquad \forall 1 < p < \infty,$$

where p and C are independent of ε. This yields the assertion (4.21).

5. Existence of a Solution to the Diffraction Problem

In this section we prove:

THEOREM 5.1. *Under the assumptions* (3.5), (3.6) *there exists a unique bounded solution u of* (3.1)–(3.4) *and, for any* $M > 1$,

$$\sum_{i=1}^{3} |u|_{C^{2+\alpha}(Q_i \cap \{|x| < M\})} \leqslant C(|\rho|_{C^{\alpha}(\mathbb{R}^2_+ \cap \{|x| < M+1\})} + |g|_{C^{1+\alpha}(\mathbb{R}^2_+ \cap \{|x| < M+1\})}), \tag{5.1}$$

where C is a constant independent of M.

We shall need the following result which is a special case of [1, Theorem 8.3]:

LEMMA 5.2. *Suppose* u_1, u_2 *belong to* $C^{2+\alpha}(\operatorname{int} \mathbb{R}^2_+) \cap C^1(\mathbb{R}^2_+)$ *and*

$$\Delta u_1 = \rho_1, \qquad \Delta u_2 = \rho_2 \qquad \text{in } \operatorname{int} \mathbb{R}^2_+,$$
$$u_1(x, 0) = u_2(x, 0),$$
$$\beta_1 \partial_y u_1(x, 0) + \beta_2 \partial_y u_2(x, 0) = g(x),$$

where β_1, β_2 *are positive constants. Then*

$$\sum_{i=1}^{2} |u_i|_{C^{2+\alpha}(\mathbb{R}^2_+)} \leqslant C\left\{\sum_{i=1}^{2} (|u_i|_{L^\infty(\mathbb{R}^2_+)} + |\rho_i|_{C^\alpha(\mathbb{R}^2_+)}) + |g|_{C^{1+\alpha}(\mathbb{R}^2_+))}\right\},$$

where C is a constant depending only on β_1, β_2.

We shall apply the lemma to

$$u_1(x, y) = u(x, y + h_1)\, \tilde{\chi}(y),$$
$$u_2(x, y) = u(x, -y + h_1)\, \tilde{\chi}(y),$$

where $\tilde{\chi}$ is defined as in the proof of Lemma 4.4. Then

$$\Delta u_1 = \rho_1 \tilde{\chi} + 2u_y \tilde{\chi}' + u\tilde{\chi}'' \equiv \tilde{\rho}_1,$$
$$\Delta u_2 = \rho_2 \tilde{\chi} + \frac{2}{\kappa_1} u_y \tilde{\chi}' + \frac{u}{\kappa_1} \tilde{\chi}'' \equiv \tilde{\rho}_2,$$
$$u_1(x, 0) = u_2(x, 0),$$
$$\partial_y u_1(x, 0) + \kappa_1 \partial_y u_2(x, 0) = g(x),$$

where $\rho_1(x, y) = -\rho(x, y+h_1)$, $\rho_2(x, y) = -(1/\kappa_1)\,\rho(x, -y+h_1)$. Applying Lemma 5.1 and estimating the C^α norm of $\tilde{\rho}_i$ by using Lemma 3.1 and the interior Schauder estimates, we conclude that

$$\sum_{i=1}^{2} |u_i|_{C^{2+\alpha}(\mathbb{R}^2_+)} < \infty.$$

This shows that u is in $C^{2+\alpha}$ in $\{h_1 - \eta \leqslant y \leqslant h_1\}$ and in $\{h_1 \leqslant y \leqslant h_1 + \eta\}$. A similar result can be established for u in $\{|y-h_2| < \eta\}$. In the remaining portions of $\mathbb{R}^2_+$, u is in $C^{2+\alpha}$ (by Schauder's estimates). Thus:

LEMMA 5.3. *The bounded solution of* (3.1)–(3.4) *(if existing) must satisfy:*

$$\sum_{i=1}^{3} |u|_{C^{2+\alpha}(\overline{Q_i})} \leqslant C\{|\rho|_{C^\alpha(\mathbb{R}^2_+)} + |g|_{C^{1+\alpha}(\mathbb{R}^1)}\}. \tag{5.2}$$

We now turn to proving the existence of a bounded solution of (3.1)–(3.4) (uniqueness was established in Corollary 3.2).

For any $R > 0$ consider the diffraction problem

$$\begin{aligned} \Delta u &= -\rho \qquad \text{in } \Omega_R, \\ [\kappa u_y]_{h_1-0}^{h_1+0} &= g(x), \\ [\kappa u_y]_{h_2-0}^{h_2+0} &= 0, \\ u(x, 0) &= 0, \end{aligned} \tag{5.3}$$

where $\Omega_R = B_R(0) \cap \mathbb{R}^2_+$, $B_R(0) = \{x^2 + y^2 < R^2\}$. By [9] there exists a unique solution u_R of the diffraction problem (5.3), and it belongs to $C^{2+\alpha}$ uniformly from each side of $y = h_1$ and from each side of $y = h_2$ (except where these lines meet $\partial\Omega_R$).

The proof of Lemma 3.1 can be modified to apply to the problem (5.3); it shows that

$$|u_R|_{L^\infty(\Omega_R)} \leqslant C(|\rho|_{L^\infty} + |g|_{L^\infty}), \tag{5.4}$$

where C is independent of R. Next, the proof of Lemma 5.3 can be modified by working with $u_i\psi(x)$ (instead of u_i) where ψ is a cut-off function. It yields the estimate

$$\begin{aligned} \sum_{i=1}^{3} |u_R|_{C^{2+\alpha}(Q_i \cap \{|x| < M\})} \leqslant C\{&|\rho|_{C^\alpha(\mathbb{R}^2_+ \cap \{|x| < M+1\})} \\ &+ |g|_{C^{1+\alpha}(\mathbb{R}^1 \cap \{|x| < M+1\})}\} \end{aligned} \tag{5.5}$$

for any $0 < M \leqslant R - 2$, where C is a constant independent of M, R.

We can now extract a sequence u_{R_j} which is convergent in $C^{2+\beta}_{\text{loc}}(\mathbb{R}^2_+ \cap Q_j)$ to a function u (for any $0<\beta<\alpha$). It is easily seen that u is a bounded solution of (3.1)–(3.4); furthermore, u satisfies (5.1). This completes the proof of Theorem 5.1.

6. An Auxiliary Parabolic Problem

In this section we consider the parabolic problem

$$\rho_t - \varepsilon \Delta \rho = \nabla u \cdot \nabla \rho - \frac{\rho^2}{\kappa} \quad \text{in} \quad \Omega_{h_1} \times (0, T), \tag{6.1}$$

$$\rho(x, y, 0) = f(x, y) \quad \text{if} \quad (x, y) \in \Omega_{h_1}, \tag{6.2}$$

$$\rho_y(x, 0, t) = \rho_y(x, h_1, t) = 0 \quad \text{if} \quad x \in \mathbb{R}^1,\ 0<t<T, \tag{6.3}$$

where

$$\begin{gathered} |f|_{L^\infty} < \infty, \qquad f \geqslant 0, \\ |\nabla u|_{L^\infty(\Omega_{h_1} \times (0, T))} < \infty. \end{gathered} \tag{6.4}$$

A function $\sigma(x, y, t)$ is said to belong to $W^{2,p}_{\text{loc}}(\Omega_{h_1} \times (0, T))$ if σ, $\nabla\sigma$, $\nabla^2\sigma$, and σ_t belong to $L^p(\Omega_{h_1} \cap K) \times (0, T)$ for any compact subset K of $\mathbb{R}^2$; here ∇ is the gradient (∂_x, ∂_y).

By a solution of (6.1)–(6.3) we mean a function ρ in $W^{2,p}_{\text{loc}}(\Omega_{h_1} \times (0, T))$ for all $1<p<\infty$ which satisfies (6.1) a.e. and (6.2), (6.3) in the usual sense (By Sobolev's imbedding, ρ is continuous up to the boundary.)

Lemma 6.1. *There exists at most one bounded solution of* (6.1)–(6.3).

Proof. Suppose ρ_1, ρ_2 are two bounded solutions and set $\rho = \rho_1 - \rho_2$. Then

$$\begin{gathered} \rho_t - \varepsilon \Delta \rho = \nabla u \cdot \nabla \rho + k\rho, \\ \rho(x, y, 0) = 0, \qquad \rho_y(x, 0, t) = \rho_y(x, h_1, t) = 0, \end{gathered}$$

where $k = -(\rho_1 + \rho_2)/\kappa$ is a bounded function. We shall compare ρ with a function

$$w(x, y) \equiv w_R(x, y) = \frac{m}{R^2}(x^2 + y^2 + Kt)\, e^{\beta t}$$

in

$$G_R \equiv (\Omega_{h_1} \cap B_R(0)) \times (0, T),$$

where β, K are positive constants and $m=|\rho|_{L^\infty}$. Then $w_R > \pm\rho$ on the parabolic boundary of G_R. Also

$$w_t - \varepsilon\Delta w - \nabla u\cdot\nabla w - kw$$

$$= \frac{m}{R^2} e^{\beta t}\{(\beta-k)(x^2+y^2+Kt)+K-4\varepsilon-2xu_x-2yu_y\}>0$$

if K and β are chosen sufficiently large (independently of R). Applying the maximum principle to $w_R \pm \rho$ in G_R we conclude that $w_R \pm \rho \geqslant 0$ in G_R, i.e.,

$$|\rho(x, y, t)| \leqslant \frac{m}{R^2}(x^2+y^2+Kt)\, e^{\beta t}.$$

Fixing (x, y, t) and letting $R \to \infty$ we deduce that $\rho(x, y, t)=0$.

COROLLARY 6.2. *If u and f are periodic in x of period a then the bounded solution $\rho(x, y, t)$ is also periodic in x of period a.*

The proof is the same as for Corollary 3.3.

THEOREM 6.3. *There exists a unique bounded solution of* (6.1)–(6.3).

Proof. By standard parabolic theory one can establish the existence of a solution ρ_R of

$$\begin{aligned}
&\rho_t - \varepsilon\Delta\rho = \nabla u\cdot\nabla\rho - \rho^2/\kappa \qquad \text{in} \quad (-R, R)\times(0, h_1)\times(0, T),\\
&\rho(x, y, 0) = f(x, y),\\
&\rho_y(x, 0, t) = \rho_y(x, h_1, t) = 0,\\
&\rho(-R, y, t) = \rho(R, y, t) = 0.
\end{aligned} \tag{6.5}$$

Indeed, this is true if ρ^2 is replaced by a monotone increasing, bounded function of ρ, say $\theta(\rho)$, and, by the maximum principle, $0 \leqslant \rho \leqslant |f|_{L^\infty}$. Approximating $(\rho^2)^+$ by a sequence of such functions $\theta(\rho)$, the existence of a solution ρ_R of (6.5) follows, and

$$0 \leqslant \rho_R \leqslant |f|_{L^\infty}. \tag{6.6}$$

By L^p-theory [8],

$$|\rho_R|_{W^{2,p}[(\Omega_{h_1}\cap\{|x|<M\})\times(0,T)]} \leqslant C\,|f|_{L^\infty}$$

for any $M>0$, $1<p<\infty$, where C is a constant depending on M, p. We can therefore extract a sequence ρ_{R_j}, with $R_j \to \infty$, which converges to a bounded solution of (6.1)–(6.3).

COROLLARY 6.4. *The bounded solution of* (6.1)–(6.3) *satisfies:*

$$0 \leqslant \rho \leqslant |f|_{L^\infty}. \tag{6.7}$$

7. SOLUTION OF (2.3)–(2.11)

In this section we prove that there exists a unique global solution of the space charge problem (2.3)–(2.11); we shall assume that the conditions (2.12)–(2.14) are satisfied.

Introduce the space

$$H = \{u(x, y) \in C(\mathbb{R}^2_+),\ \|u\|^* < \infty, \text{ and } u \text{ is periodic in } x \text{ of period a}\},$$

where

$$\|u\|^* = \sum_{i=1}^{3} |u|_{C^{2+\alpha}(Q_i)}.$$

Set

$$S^{t_0} = \{u \in C^0([0, t_0], H);\ \sup_{0 \leqslant t \leqslant t_0} \|u(t)\|^* < \infty\}.$$

Then S^{t_0} is a Banach space with the norm

$$\|u\| = \sup_{0 \leqslant t \leqslant t_0} \|u(t)\|^*.$$

For any $M > 0$ let $S^{t_0}_M = \{u \in S^{t_0};\ \|u\| \leqslant M\}$.

For any $u \in S^{t_0}_M$ denote by ρ the bounded solution of (6.1)–(6.3) (with $T = t_0$) established in Theorem 6.3. By Corollary 6.2, ρ is periodic in x of period a. Let $\tilde{u}$ denote the bounded solution of (3.1)–(3.4) corresponding to

$$g(x) = \int_0^t \rho(x, h_1, s)\, u_y(x, h_1 - 0, s)\, ds - \sigma_0(x).$$

By Corollary 3.3, ũ is periodic in x of period a. Define a mapping L by $\tilde{u} = Lu$.

Set $D = [0, a] \times [0, h_1]$. From Lemma 5.3 we get

$$\begin{aligned} \|\tilde{u}(t)\|^* &\leqslant C(|\rho(t)|_{C^\alpha(\mathbb{R}^2_+)} + |g|_{C^{1+\alpha}(\mathbb{R}^1)}) \\ &\leqslant C\{1 + |\rho(t)|_{C^\alpha(D)} + t_0 \sup_{0 \leqslant s \leqslant t_0} |\rho(s)|_{C^{1+\alpha}(D)} \cdot M\}. \end{aligned} \tag{7.1}$$

We shall need the following Sobolev imbedding result (see [8, Chap. 2, Lemma 3.3]):

LEMMA 7.1. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ satisfying the cone condition. If $p > n+2$ and $\alpha = 1 - ((n+2)/p)$ then*

$$|w|_{C^\alpha(\Omega_T)} + |D_x w|_{C^\alpha(\Omega_T)} \leqslant C\,|w|_{W^{2,p}(\Omega_T)}$$

for any $w \in W^{2,p}(\Omega_T)$, where $\Omega_T = \Omega \times (0, T)$.

Here

$$|u|_{C^\alpha(\Omega_T)} = |u|_{L^\infty(\Omega_T)} + \sup_{P,\, Q \in \Omega_T} \frac{|u(P) - u(Q)|}{d(P, Q)^\alpha},$$

where $P = (x, t)$, $Q = (\bar{x}, \bar{t})$, $P \neq Q$, and

$$d(P, Q) = \{|x - \bar{x}|^2 + |t - \bar{t}|\}^{1/2}:$$

The $W^{2,p}(\Omega_T)$-norm of w is the sum of the L^p norms of w, w_t, $D_x w$, and $D_x^2 w$.

By L^p-estimates applied to (6.1)–(6.3), and the periodicity of all the functions occurring in (6.1)–(6.3),

$$|\rho|_{W^{2,p}(D \times (0,T))} \leqslant \sigma(M), \tag{7.2}$$

where $\sigma(M)$ is a monotone increasing finite-valued function of M. Applying Lemma 7.1 to ρ we conclude that

$$|\rho|_{C^{1+\alpha}(D \times (0, t_0))} \leqslant C\,|\rho|_{W^{2,p}(D \times (0, t_0))} \leqslant \sigma(M), \tag{7.3}$$

where

$$|\rho|_{C^{1+\alpha}(D \times (0, t_0))} = |\rho|_{C^\alpha(D \times (0, t_0))} + |\nabla\rho|_{C^\alpha(D \times (0, t_0))}. \tag{7.4}$$

Since

$$\begin{aligned} |\nabla\rho(x, y, t)| &\leqslant |\nabla\rho(x, y, t) - \nabla\rho(x, y, 0)| + |\nabla f(x, y)| \\ &\leqslant |\nabla\rho|_{C^\alpha} t^{\alpha/2} + |\nabla f|_{L^\infty}, \end{aligned}$$

we get

$$\begin{aligned} |\rho(t)|_{C^\alpha(D)} &\leqslant |\nabla\rho(t)|_{L^\infty(D)} + C\,|\rho(t)|_{L^\infty(D)} \\ &\leqslant t^{\alpha/2}\,|\nabla\rho|_{C^\alpha(D \times (0, t_0))} + |\nabla f|_{L^\infty} + C\,|f|_{L^\infty} \\ &\leqslant C + t_0^{\alpha/2}\sigma(M) \qquad \text{(by (7.3))}. \end{aligned} \tag{7.5}$$

Using (7.3), (7.5) in (7.1), we obtain

$$\sup_{0 \leqslant t \leqslant t_0} \|\tilde{u}(t)\|^* \leqslant C + (t_0 + t_0^{\alpha/2})\,\sigma(M). \tag{7.6}$$

Hence, choosing $M = C + 1$ and t_0 sufficiently small we conclude that $\tilde{u}$ belongs to $S_M^{t_0}$, i.e., L maps $S_M^{t_0}$ into itself.

Next we show that L is a contraction if t_0 is sufficiently small.

Let u_1, u_2 belong to $S_M^{t_0}$ and denote the corresponding ρ, $\tilde{u}$ by ρ_1, $\tilde{u}_1$ and ρ_2, $\tilde{u}_2$. Set $\rho = \rho_1 - \rho_2$, $\tilde{u} = \tilde{u}_1 - \tilde{u}_2 = Lu_1 - Lu_2$. By Lemma 7.1,

$$|\rho|_{C^{1+\alpha}(D \times (0, t_0))} \leqslant C\, |\rho|_{W^{2,p}(D \times (0, t_0))}. \tag{7.7}$$

Applying L^p-estimates to

$$\rho_t - \varepsilon \Delta \rho - \nabla u_1 \cdot \nabla \rho + k\rho = \nabla \rho_2 \cdot \nabla(u_1 - u_2), \qquad k = \frac{\rho_1 + \rho_2}{\kappa},$$

we deduce that the right-hand side of (7.7) is bounded by

$$\sigma(M)\, |\nabla(u_1 - u_2)|_{L^\infty(D \times (0, t_0))}.$$

Consequently

$$|\rho|_{C^{1+\alpha}(D \times (0, t_0))} \leqslant \sigma(M)\, \|u_1 - u_2\|, \tag{7.8}$$

where $\sigma(M)$ is a constant times the original function $\sigma(M)$. Since $\rho(x, y, 0) = 0$,

$$\begin{aligned}
|\rho(t)|_{C^\alpha(D)} &\leqslant |\nabla \rho(t)|_{L^\infty(D)} + C\, |\rho(t)|_{L^\infty(D)} \\
&\leqslant t^{\alpha/2}\, |\nabla \rho|_{C^\alpha(D \times (0, t_0))} + C t^{\alpha/2}\, |\rho|_{C^\alpha(D \times (0, t_0))} \\
&\leqslant C t_0^{\alpha/2}\, |\rho|_{C^{1+\alpha}(D \times (0, t_0))} \leqslant C\sigma(M)\, t_0^{\alpha/2}\, \|u_1 - u_2\|,
\end{aligned} \tag{7.9}$$

where (7.8) was used. If we now apply Lemma 5.3 we obtain

$$\begin{aligned}
\|Lu_1(t) - Lu_2(t)\|^* = \|\tilde{u}(t)\|^* &\leqslant C(|\rho(t)|_{C^\alpha(D)} + |g|_{C^{1+\alpha}(0, 1)}) \\
&\leqslant C\sigma(M)(t_0^{\alpha/2} + t_0)\, \|u_1 - u_2\|,
\end{aligned}$$

since

$$g(x) = \int_0^t [\rho(x, h_1, s)\, \partial_y u_1(x, h_1 - 0, s) - \rho_2(x, h_1, s)\, \partial_y \tilde{u}(x, h_1 - 0, s)]\, ds.$$

We conclude that L is a contraction in $S_M^{t_0}$ if t_0 is sufficiently small. Hence L has a unique fixed point V in $S_M^{t_0}$.

By a solution of the problem (2.3)–(2.11) we mean a pair (ρ, V) with $V \in W_{\text{loc}}^{2,p}$ for any $1 < p < \infty$ such that (2.3) holds a.e., and all other equations and boundary and initial conditions are satisfied in the classical sense. It follows, in particular, that V must belong to S^{t_0}. If (ρ', V') is another solution for $0 \leqslant t \leqslant t^0$, then V' must belong to $S_{M'}^{t_0}$ for some M' sufficiently large. Working with M' (instead of M) we deduce that L is a contraction if t_0 is replaced by some smaller $t' \in (0, t_0)$. Hence $V = V'$ if $0 \leqslant t \leqslant t'$ and, proceeding step-by-step, we find that $V = V'$ for all $0 \leqslant t \leqslant t_0$. We have thus proved:

LEMMA 7.2. *There exists a unique solution of* (2.3)–(2.11) *for* $0 \leqslant t \leqslant t_0$, *provided* t_0 *is sufficiently small.*

Notice that (2.9) is satisfied by Lemma 4.3. In order to extend the solution globally, we need some a priori estimates. Thus we shall now assume that the solution exists for all $0 \leqslant t < T$ ($t_0 \leqslant T < \infty$) and derive a priori estimates.

First recall that by the maximum principle

$$0 \leqslant \rho \leqslant |f|_{L^\infty}. \tag{7.10}$$

Next we prove:

LEMMA 7.3. *There holds*

$$\sup_{0 \leqslant x \leqslant a} |V_y(x, h_1 - 0, t)| \leqslant C e^{ct}, \tag{7.11}$$

where C, c *are positive constants independent of* T *and* ε.

Proof. By Lemma 4.4,

$$|V_y(x, h_1 - 0 < t)| \leqslant C \left\{ |\rho|_{L^\infty} + \left| \int_0^t \rho(x, h_1, s)\, V_y(x, h_1 - 0, s)\, ds - \sigma_0(x) \right|_{L^\infty} \right\}$$
$$\leqslant C\, |\rho|_{L^\infty} \left(1 + \int_0^t \sup_{0 \leqslant x \leqslant a} |V_y(x, h_1 - 0, s)|\, ds \right).$$

Applying Gronwall's inequality and recalling (7.10), the assertion (7.11) follows.

LEMMA 7.4. *For any* $p > 1$, $0 < t < T$,

$$\sum_{i=1}^{3} |\nabla V(t)|_{L^p\{Q_i \cap [0 \leqslant x \leqslant a]\}} \leqslant C, \tag{7.12}$$

where C *is a constant independent of* ε, t.

Proof. From Lemmas 4.4, 4.5, and 7.3,

$$|V_y(\cdot, t)|_{L^\infty(\mathbb{R}^2_+)} \leqslant C \tag{7.13}$$

and

$$|V_x(x, y, t)| \leqslant C(1 + |\log|y - h_1||), \tag{7.14}$$

where C is a constant independent of t, ε. The assertion (7.12) now immediately follows.

We can now apply L^p-estimates to the parabolic equation (2.3), making use of the fact $\nabla V \in L^p$ for any $p > 1$ (see [8, Chap. IV, Sect. 9]). We deduce that

$$|\rho|_{W^{2,p}(D \times (0,T))} \leqslant C_\varepsilon, \tag{7.15}$$

where C_ε is a constant depending on ε and on the constant C in (7.12). By estimate (5.1) of Theorem 5.1 we then obtain

$$\begin{aligned}
\|V(t)\|^* &\leqslant C\left(|\rho(t)|_{C^\alpha(D)} + \left|\int_0^t \rho(x, h_1, s)\, V_y(x, h_1 - 0, s)\, ds - \sigma_0\right|_{C^{1+\alpha}}\right) \\
&\leqslant C_\varepsilon\, |\rho(t)|_{C^{1+\alpha}(D)} \left(1 + \int_0^t \|V(s)\|^*\, ds\right) \\
&\leqslant C_\varepsilon \left(1 + \int_0^t \|V(s)\|^*\, ds\right), \qquad \text{by (7.15).}
\end{aligned}$$

By Gronwall's inequality we then get the a priori bound

$$\|V(t)\|^* \leqslant C_\varepsilon e^{C_\varepsilon t}, \qquad 0 \leqslant t < T. \tag{7.16}$$

This estimate can be used to extend the solution beyond $t = T$. Indeed, we can repeat the proof of Lemma 7.2 with $t = T - \delta$ as the initial time; in view of (7.16) the solution exists for $T - \delta \leqslant t \leqslant T - \delta + t_0$ where t_0 is independent of δ (indeed, t_0 depends on how large M is, whereas M is independent of δ if $\|V(t-\delta)\|^* \leqslant C$). It follows that the solution can be extended to $0 \leqslant t < T + t_0$.

Proceeding step-by-step, we can extend the solution to all $t > 0$. Thus we have proved:

THEOREM 7.5. *There exists a unique solution* $(\rho_\varepsilon, V_\varepsilon)$ *of* (2.3)–(2.11) *for all* $0 \leqslant t < \infty$. *The solution satisfies* (7.10); *further, for any* $T > 0$, (7.13), (7.14) *hold for all* $0 < t < T$, *where* C *is a constant depending on* T, *but independent of* ε.

8. THE VECTOR FIELD ∇V_ε FOR LARGE $|\sigma_0|$

The rest of the paper is devoted to establishing that in some sense positive charge is deposited on $y = h_1$, uniformly in ε. More precisely, if we set

$$\Sigma_\varepsilon = \{(x, t); 0 < x < a, 0 < t < T, \rho_\varepsilon(x, h_1, t) > c\} \tag{8.1}$$

then, we would like to prove that for some $T>0$, $c>0$, $c_*>0$,

$$\text{meas } \Sigma_\varepsilon > c_* \qquad \text{for all small } \varepsilon > 0. \tag{8.2}$$

In Section 9 we shall establish a weak version of this property provided $|\sigma_0|$ is sufficiently large. For simplicity we assume that

$$\sigma_0(x) = \text{const.} = \sigma_0, \tag{8.3}$$

$$0 \leqslant f(x, y) \leqslant 1. \tag{8.4}$$

We shall also need the physical assumption that

$$\sigma_0 < 0, \qquad |\sigma_0| \text{ is sufficiently large.} \tag{8.5}$$

In this section we prove:

THEOREM 8.1. *For any $0<\delta<1$ there exist positive constants T and R_0 such that*

$$-A_0\,|\sigma_0| \leqslant \partial_y V_\varepsilon \leqslant -A\,|\sigma_0| \qquad \text{in} \quad \Omega_{h_1} \times (0, T), \tag{8.6}$$

$$|\partial_x V_\varepsilon| \leqslant \delta\,|\sigma_0|\,(1 + |\log|h_1 - y||) \qquad \text{in} \quad \Omega_{h_1} \times (0, T) \tag{8.7}$$

provided $|\sigma_0| \geqslant R_0$, where

$$T = \frac{\delta}{A_1}, \qquad R_0 = \frac{A_2}{\delta}, \qquad 2A = \min\left\{\frac{1}{\kappa_1}, \frac{1}{\kappa_2}\right\},$$

and A_0, A_1, A_2 are positive constants depending only on κ_1, κ_2, h_1, h_2.

Proof. Set $R = |\sigma_0|$ and consider the function $w_R = V_\varepsilon/R$. It satisfies

$$\begin{aligned}
&\nabla \cdot (\kappa \nabla w_R) = -\frac{\rho_\varepsilon}{R} \qquad \text{in} \quad \mathbb{R}^2_+ \setminus (\{y = h_1\} \cup \{y = h_2\}),\\
&w_R(x, 0) = 0,\\
&[\kappa \partial_y w_R]_{h_2 - 0}^{h_2 + 0} = 0,\\
&[\kappa \partial_y w_R]_{h_1 - 0}^{h_1 + 0} = 1 + \int_0^t \rho_\varepsilon(x, h_1, s)\, w_{R,y}(x, h_1 - 0, s)\, ds.
\end{aligned} \tag{8.8}$$

In the sequel we denote various positive constants depending only on the κ_i, h_i by C. By Lemmas 4.4, 4.5,

$$\begin{aligned}
|\partial_y V_\varepsilon|_{L^\infty} &\leqslant C\left(|\rho_\varepsilon|_{L^\infty} + \left|R + \int_0^t (\rho_\varepsilon V_{\varepsilon,y})(x, h_1 - 0, s)\, ds\right|_{L^\infty}\right)\\
&\leqslant C\left(1 + R + \int_0^t |V_{\varepsilon,y}(\cdot, s)|_{L^\infty}\, ds\right),
\end{aligned}$$

so that, by Gronwall's inequality,

$$|\partial_y V_\varepsilon|_{L^\infty} \leqslant CR \tag{8.9}$$

if $R \geqslant 1$, $t \leqslant 1$.

By (3.7), (8.9),

$$\begin{aligned}|V_\varepsilon|_{L^\infty} &\leqslant C\left(1 + \left| R + \int_0^t (\rho_\varepsilon V_{\varepsilon, y})(x, h_1 - 0, s)\, ds \right|_{L^\infty}\right)\\ &\leqslant C\left(1 + R\left(1 + \int_0^t C\, ds\right)\right)\end{aligned}$$

so that

$$|V_\varepsilon|_{L^\infty} \leqslant CR \tag{8.10}$$

if $t \leqslant 1$, and

$$|w_R(\cdot, t)|_{L^\infty} \leqslant C. \tag{8.11}$$

Since, for each ε, w_R is a bounded solution of (8.8) (uniformly in R), Lemma 5.3 yields

$$\sum_{i=1}^3 |w_R|_{C^{2+\alpha}(Q_i)} \leqslant C_\varepsilon \left\{\frac{1}{R} |\rho_\varepsilon|_{C^\alpha} + 1 + \int_0^t |w_R|_{C^{2+\alpha}}\, ds\right\}$$

and, then, by Gronwall's inequality,

$$\sum_{i=1}^3 |w_R|_{C^{2+\alpha}(Q_i)} \leqslant C_\varepsilon$$

if $t < 1$. From any sequence w_R $(R \to \infty)$ we can therefore extract a subsequence w_{R_n} such that $w_{R_n} \to w_0$ in $C^{2+\beta}(Q_i)$ for any $0 < \beta < \alpha$, and w_0 is a solution of

$$\begin{aligned} &\nabla \cdot (\kappa \nabla w_0) = 0 \qquad \text{in} \quad \mathbb{R}^2_+ \backslash (\{y = h_1\} \cup \{y = h_2\}),\\ &w_0(x, 0) = 0,\\ &[\kappa \partial_y w_0]_{h_2 - 0}^{h_2 + 0} = 0\\ &[\kappa \partial_y w_0]_{h_1 - 0}^{h_1 + 0} = g(x, t), \end{aligned} \tag{8.12}$$

where

$$g(x, t) = 1 + \int_0^t \rho_\varepsilon(x, h_1, s)\, w_{0, y}(x, h_1 - 0, s)\, ds. \tag{8.13}$$

Since w_0 is a bounded solution of (8.12), it is also unique, by Corollary 3.2. Consequently, as $R \to \infty$,

$$w_R \to w_0 \quad \text{in} \quad C^{2+\beta}(Q_i)\ \forall \beta < \alpha. \tag{8.14}$$

Applying Lemmas 4.4, 4.5 to w_0 and using Gronwall's inequality as before we find that

$$|\partial_y w_0|_{L^\infty} \leqslant C \tag{8.15}$$

and therefore

$$\left| \int_0^t \rho_\varepsilon(x, h_1, s)\, w_{0,y}\,(x, h_1 - 0, s)\, ds \right| \leqslant Ct. \tag{8.16}$$

Let $\tilde{w}$ denote the bounded solution of (8.12) when $g \equiv 1$. One can immediately verify that

$$\tilde{w} = \begin{cases} -\dfrac{1}{\kappa_2}\, y & \text{if} \quad 0 \leqslant y \leqslant h_2 \\[2ex] -\dfrac{1}{\kappa_1}\, y + \left(\dfrac{1}{\kappa_1} - \dfrac{1}{\kappa_2}\right) h_2 & \text{if} \quad h_2 \leqslant y \leqslant h_1 \\[2ex] -\dfrac{1}{\kappa_1}\, h_1 + \left(\dfrac{1}{\kappa_1} - \dfrac{1}{\kappa_2}\right) h_2 & \text{if} \quad h_1 \leqslant y < \infty. \end{cases} \tag{8.17}$$

Therefore

$$\tilde{w}_y \leqslant \max\left\{ -\frac{1}{\kappa_2}, -\frac{1}{\kappa_1} \right\} \equiv -A^* < 0 \quad \text{in} \quad \{0 \leqslant y \leqslant h_1\}. \tag{8.18}$$

The function $W = w_0 - \tilde{w}$ satisfies:

$$\begin{aligned} &\nabla \cdot (\kappa \nabla W) = 0 \qquad \text{in } \mathbb{R}^2_+ \setminus (\{y = h_1\} \cup \{y = h_2\}), \\ &[\kappa W_y]_{h_2 - 0}^{h_2 + 0} = 0, \\ &[\kappa W_y]_{h_1 - 0}^{h_1 + 0} = \int_0^t \rho_\varepsilon(x, h_1, s)\, w_{0,y}(x, h_1 - 0, s)\, ds, \\ &W(x, 0) = 0. \end{aligned}$$

Applying Lemmas 4.4, 4.5 and using (8.16), we get

$$|W_y|_{L^\infty} \leqslant Ct, \tag{8.19}$$

$$|W_x|_{L^\infty} \leqslant Ct(1 + |\log|y - h_1||). \tag{8.20}$$

Recalling (8.18) we conclude that

$$-C \leqslant \partial_y w_0 \leqslant -\frac{3A^*}{4} \qquad \text{in} \quad \Omega_{h_1} \times (0, T), \tag{8.21}$$

$$|\partial_x w_0| \leqslant Ct(1 + |\log|y - h_1||) \qquad \text{in} \quad \Omega_{h_1} \times (0, T) \tag{8.22}$$

provided $T \leqslant A^*/2C$.

In order to complete the proof of the lemma we estimate $u \equiv w_R - w_0$. Since u is the bounded solution of (8.8) with the last equation replaced by

$$[\kappa \partial_y u]_{h_1 - 0}^{h_1 + 0} = \int_0^t \rho_\varepsilon(x, h_1, s)\, u_y(x, h_1 - 0, s)\, ds,$$

Lemmas 4.4 and 4.5 give

$$|u_y(\cdot, t)|_{L^\infty} \leqslant C\left(\frac{1}{R} + \int_0^t |u_y(\cdot, s)|_{L^\infty}\, ds\right),$$

or

$$|u_y(\cdot, t)|_{L^\infty} \leqslant \frac{C}{R}. \tag{8.23}$$

Also by the same lemmas and (8.23),

$$|u_x(\cdot, t)| \leqslant \frac{C}{R}(1 + |\log|y - h_1||). \tag{8.24}$$

Recalling (8.21), (8.22) the assertions (8.6), (8.7) readily follow.

9. Uniform Positivity of ρ_ε near $y = h_1$

In this section we establish (8.2) in some weak sense. Suppose

$$f(x, y) > \gamma > 0 \qquad \text{if} \quad a_1 \leqslant x \leqslant a_2,\ 0 \leqslant y \leqslant b_1, \tag{9.1}$$

for some $0 < a_1 < a_2 < a$, $0 < b_1 < h_1$.

Set $a_0 = \frac{1}{2}(a_1 + a_2)$ and let $f_0(y)$ be a smooth function satisfying:

$$\begin{aligned} &f_0(y) = \gamma && \text{if} \quad 0 \leqslant y \leqslant b_1, \\ f \geqslant f_0 \geqslant 0 \qquad &f_0'(y) \leqslant 0 && \text{if} \quad y \geqslant b_1. \end{aligned} \tag{9.2}$$

Set

$$R = |\sigma_0|, \qquad \phi(y) = \delta R(1 + |\log|y - h_1||) \tag{9.3}$$

and denote by p_ε the solution of

$$p_t - \varepsilon \Delta p + \operatorname{sgn}(a_0 - x)\,\phi(y)\,p_x + ARp_y + \mu p = 0$$

$$\text{in } (a_1, a_2)(0, h_1) \times (0, T) \qquad \left(\mu \geqslant \frac{1}{\kappa_i}\ \forall i\right), \tag{9.4}$$

$$p(a_1, y, t) = p(a_2, y, t) = 0, \tag{9.5}$$

$$p_y(x, 0, t) = p_y(x, h_1, t) = 0, \tag{9.6}$$

$$p(x, y, 0) = f_0(y). \tag{9.7}$$

LEMMA 9.1. *If $R \geqslant R_0$ then*

$$\rho_\varepsilon(x, y, t) \geqslant p_\varepsilon(x, y, t) \qquad \text{in } (a_1, a_2) \times (0, h_1) \times (0, T).$$

Proof. By the maximum principle

$$0 \leqslant p_\varepsilon \leqslant \gamma.$$

Since $p_\varepsilon(a_i, y, t) = 0$ we get

$$p_{\varepsilon,x}(a_1, y, t) \geqslant 0, \qquad p_{\varepsilon,x}(a_2, y, t) \leqslant 0.$$

The reflection $\tilde{p}(x, y, t) \equiv p_\varepsilon(2a_0 - x, y, t)$ of p_ε with respect to $x = a_0$ is also a solution of (9.4)–(9.7) so that $p_\varepsilon(2a_0 - x, y, t) = p_\varepsilon(x, y, t)$, by uniqueness. It follows that

$$p_{\varepsilon,x}(a_0, y, t) = 0. \tag{9.8}$$

The function $q = p_{\varepsilon,x}$ satisfies

$$q_t - \varepsilon \Delta q + \operatorname{sgn}(a_0 - x)\,\phi(y)\,q_x + ARq_y + \mu q = 0$$

in $a_1 < x < a_0$ and by the maximum principle it then follows that $q \geqslant 0$, i.e.,

$$p_{\varepsilon,x}(x, y, t) \geqslant 0 \qquad \text{if} \quad a_1 < x < a_0. \tag{9.9}$$

Similarly

$$p_{\varepsilon,x}(x, y, t) < 0 \qquad \text{if} \quad a_0 < x < a_2. \tag{9.10}$$

Next, the function $z = p_{\varepsilon,y}$ satisfies

$$z_t - \varepsilon \Delta z + \operatorname{sgn}(a - x)\,\phi(y)\,z_x + ARz_y + \mu z = -\phi'(y)\operatorname{sgn}(a_0 - x)\,p_{\varepsilon,x} \leqslant 0$$

by (9.9), (9.10) and the fact that $\phi'(y) > 0$. Applying the maximum principle we deduce that $z \leqslant 0$, i.e.,

$$p_{\varepsilon, y} \leqslant 0. \tag{9.11}$$

Using (9.8)–(9.11) and (8.6), (8.7) we find that

$$\begin{aligned} &p_{\varepsilon,t} - \varepsilon \Delta p_\varepsilon - V_{\varepsilon,x} p_{\varepsilon,x} - V_{\varepsilon,y} p_{\varepsilon,y} + p_\varepsilon^2/\kappa \\ &\quad \leqslant p_{\varepsilon,t} - \varepsilon \Delta p_\varepsilon + \operatorname{sgn}(a_0 - x)\, \phi(y)\, p_{\varepsilon,x} + ARp_{\varepsilon,y} + \mu p_\varepsilon = 0. \end{aligned}$$

Also

$$p_\varepsilon(a_i, y, t) = 0 \leqslant \rho_\varepsilon(a_i, y, t),$$

and

$$p_{\varepsilon,y} = 0 = \rho_{\varepsilon,y} \qquad \text{on} \quad y = 0,\ y = h_1.$$

Hence, by comparison, $p_\varepsilon \leqslant \rho_\varepsilon$.

We proceed to study the behavior of p_ε in $\{a_1 < x < a_0\}$ near $y = h_1$, using the boundary layer theory. We begin by considering the reduced equation

$$v_t + \phi(y)\, v_x + ARv_y + \mu v = 0. \tag{9.12}$$

The characteristic curves are given by

$$\frac{dx}{dt} = \phi(y), \qquad \frac{dy}{dt} = AR, \tag{9.13}$$

or

$$\begin{aligned} x(t) &= x_0 + \int_0^t \phi(y(s))\, ds, \\ y(t) &= y_0 + ARt \qquad (a_1 < x_0 < a_2,\ 0 < y_0 < h_1). \end{aligned} \tag{9.14}$$

We are interested in those initial points (x_0, y_0) for which the corresponding characteristics remain in

$$G \equiv (a_1, a_0) \times (0, h_1) \times (0, T)$$

and $y(t_0) = h_1$, for some $0 < t_0 \leqslant T$.

Assuming $y(t_0) = h_1$ we get

$$t_0 = \frac{h_1 - y_0}{AR} \tag{9.15}$$

and

$$x(t_0) = x_0 + \int_0^{t_0} \phi(y(t))\,dt = x_0 + \int_{y_0}^{h_1} \phi(y)\,\frac{dy}{AR}$$

$$\leqslant x_0 + \delta \int_0^{h_1} [1 + |\log(h_1 - y)|]\,dy \equiv x_0 + \delta c_0 \qquad (c_0 = c_0(h_1) > 0).$$

Hence

LEMMA 9.2. *For any* $0 < c < \frac{1}{2}(a_0 - a_1)$, *if* $\delta < c/c_0$ *then, whenever* $x(0) = x_0 \in (a_1, a_1 + c)$,

$$a < x(t) < a_1 + 2c < a_0 \qquad \text{if} \quad 0 \leqslant t \leqslant t_0,$$

where t_0 *is defined by* (9.15); *in particular, the characteristic curve corresponding to* (x_0, y_0) *remains in* G *for* $0 \leqslant t \leqslant t_0$, *and* $y(t_0) = h_1$.

Observe that the condition $t_0 < T$ is satisfied if

$$\frac{h_1}{AR_0} < T;$$

by the definitions of T and R_0 in Lemma 8.1 this inequality holds if A_2 is chosen $\geqslant h_1 A_1 / A$.

Set

$$\tilde{\Omega}_\lambda = \{(x, y, t);\, a_1 < x < a_0,\, 0 < y < \lambda,\, 0 < t < T\}.$$

For any $\eta > 0$ let $p_{\varepsilon,\eta}$ denote the solution of (9.4) in $\tilde{\Omega}_{h_1 - \eta}$ with the boundary condition

$$p(a_1, y, t) = 0, \qquad p_x(a_0, y, t) = 0,$$
$$p_y(x, 0, t) = 0, \qquad p(\lambda, h_1 - \eta, t) = 0,$$
$$p(x, y, t) = \tilde{f}(x, y),$$

where $\tilde{f}$ is a smooth function satisfying: $\tilde{f} \leqslant f_0$ and

$$\tilde{f}(x, y) = \begin{cases} \gamma & \text{if} \quad a_1 + \eta_1 < x < a_1 + \frac{1}{3}(a_0 - a_1),\ \eta_1 < y < b_1 \\ 0 & \text{if} \quad x < a_1 + \frac{1}{2}\eta_1 \text{ or } x > a_1 + \frac{1}{2}(a_0 - a_1) \text{ or } y < \frac{1}{2}\eta_1; \end{cases}$$

here η_1 is any positive number, smaller than $\frac{1}{4}(a_0 - a_1)$ and $b_1/2$.

Since $p_{\varepsilon,\eta} \geqslant 0 = p_{\varepsilon,\eta}$ on $y = h_1 - \eta$, we have, by comparison,

$$p_\varepsilon(x, y, t) \geqslant p_{\varepsilon,\eta}(x, y, t). \tag{9.16}$$

The solution v of (9.12) with $v(x, y, 0) = \tilde{f}(x, y)$, $v_y(x, 0, t) = 0$, $v(a_1, y, t) = 0$ satisfies

$$e^{\mu t} v(x(t), y(t), t) = \text{const.}$$

along characteristics. By Lemma 9.2, $v_x(a_0, y, t) = 0$ and

$$e^{\mu t} v(x(t), y(t), t) = \gamma \qquad \text{if} \quad a_1 + \eta_1 < x(0) < a_1 + \tfrac{1}{3}(a_0 - a_1) \quad (9.17)$$

(the corresponding characteristics remain in $\tilde{\Omega}_{h_1}$ for all $0 < t < t_0$, t_0 as in (9.15)), and v is smooth due to the special choice of $\tilde{f}$.

Since $\phi(y)$ is unbounded as $y \to h_1$, we cannot apply the boundary layer theory [2, 3] to p_ε. However, we can apply it to $p_{\varepsilon,\eta}$ and deduce that

$$p_{\varepsilon,\eta} = v + U_\varepsilon + w_\varepsilon,$$

where U_ε is a boundary layer which converges to zero exponentially with ε in every compact subset of $\tilde{\Omega}_{h_1-\eta}$, and $|w_\varepsilon| \leqslant C\varepsilon^{1/2}$.

Recalling Lemma 9.1, (9.16), and (9.17) we deduce:

THEOREM 9.3. *For any* $0 < \gamma_0 < \gamma$, *if* $a_1 + \eta_1 < x(0) < a_1 + \frac{1}{3}(a_0 - a_1)$, $\eta_1 < y(0) < b_1$ *then*

$$\rho_\varepsilon(x(t), y(t), t) \geqslant \gamma_0 e^{-\mu t} \tag{9.18}$$

as long as $0 \leqslant t \leqslant t_0 - \delta_0$ *where* δ_0 *is any given positive number and* ε *is sufficiently small (depending on* δ_0*).*

Since ε is actually a very small parameter, one is interested in the limits

$$\rho_0 = \lim \rho_\varepsilon, \qquad V_0 = \lim V_\varepsilon;$$

the first limit is taken in the $(L^\infty)^*$-topology. The pair (ρ_0, V_0) can be viewed as a weak solution of (2.3)–(2.12) when $\varepsilon = 0$.

Theorem 8.1 gives information on the field ∇V_0, and Theorem 9.3 gives:

COROLLARY 9.4. *For almost all* $x(0) \in (a_1 + \eta_1, a_1 + \frac{1}{3}(a_0 - a_1))$, $y(0) \in (\eta_1, b_1)$, *and* $0 \leqslant t \leqslant t_0$,

$$\rho_0(x(t), y(t), t) \geqslant e^{-\mu t}.$$

Physical Interpretation. This corollary shows that (for $\varepsilon = 0$) positive charge is in fact deposited on $y(t) = h_1$ in, say, the average sense. The deposition occurs above any x-interval where $f(x, y)$ is strictly positive. The photoelectrographic interpretation is that if we have initial good enough field in the sense that there is sufficient exposure of a large enough area in the document, then we get a good crisp image provided the surface charge σ_0 is negative and sufficiently large in absolute value. The image tends to spread a bit; for instance if f is as in (1.14) then the spread lies in $a_1 - \eta < x < a_2 + \eta$ where $\eta \leqslant C/|\sigma_0|$ (as seen from the explicit form of the solution v to (9.12)).

We can establish the stronger version (8.2) only in the special case where

$$f(x, y) \text{ is independent of } x.$$

In this case ρ_ε and p_ε are independent of x and the boundary layer theory [2, 3] gives:

$$p_\varepsilon(x, t) = v + O(\varepsilon).$$

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